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ANALYTICITY OF NONLINEAR SEMIGROUPS

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ABSTRACT
The Cauchy problem $du/dt = Au(t)$, $u(0) = u_0 \in D(A)$ has analytic solutions when A has first and second Gateaux derivatives along the solution curve in a certain weak sense. Here A is a maximal monotone operator in a complex Hilbert space.

0. Introduction

In this paper we discuss nonlinear holomorphic semigroups in Hilbert spaces, intending to remove the assumption of the complex Fréchet differentiability of "resolvent" of $A : (\lambda I - A)^{-1}$.

K. Yosida [11] first established analyticity of semigroups of linear operators. T. Kato–H. Tanabe [5] and K. Masuda [9] considered linear holomorphic evolution operators. In case of semilinear and quasilinear equations several authors discussed analyticity of the solutions (S. Ōuchi [10], Hayden–Massey [4], Massey [8], Furuya [2, 3]).

Y. Kōmura [6] gave the relation of nonlinear holomorphic semigroups to resolvents of generators. Instead of linearity of A, he assumed complex Fréchet differentiability of the resolvent of A.

In the present paper we assume only "temporal analyticity" of A, more precisely we consider analyticity for the equation $du(t)/dt \in Au(t)$ under the assumption that A has first and second Gateaux derivatives along the solution curve in a certain weak sense.

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1. Main theorem

We establish analyticity in t of solutions to nonlinear evolution equations:

$$\frac{d}{dt}u(t) = Au(t), \qquad 0 \leqq t \leqq T, \tag{1.1}$$

$$u(0) = u_0 \in D(A). \tag{1.2}$$

Let $(H, \|\cdot\|)$ be a complex Hilbert space with norm $\|\cdot\|$.

We assume the following conditions:

(A.1) A is a single valued maximal monotone operator from H to H.

(A.2) G is a normed space with norm $\|\cdot\|_G$. G contains H and $\|\cdot\|_G \leqq \|\cdot\|$.

(A.3) There is a single valued operator A_G from H to G with restriction $A_G|_{D(A)} = A$ and $D(A_G) = H$.

(A.4) For any $x \in D(A)$ there exist a linear function $\partial A(x)$ from H to G, $L > 0$, and $\alpha(x) > 0$ satisfying the following:

$$A_G(x + \lambda Ax + \varepsilon(\lambda)) = Ax + \lambda \partial A(x)Ax + \eta(\lambda), \tag{1.3}$$

$$\|\partial A(x)Ax\|_G \leqq L\|Ax\| \qquad \text{for } |\lambda| \leqq \alpha(x), \quad \lambda \in \mathbf{C},$$

$$\|\eta(\lambda)\|_G \leqq H(\lambda) = o(|\lambda|), \qquad \|\varepsilon(\lambda)\| \leqq E(\lambda) = o(|\lambda|),$$

$$\inf\{\alpha(x); \|x\| < N\} = \alpha_N > 0.$$

(A.5) There exist a constant $\omega \in (0, \pi/2)$ and a "resolvent" operator $(A + \lambda)^{-1}$ satisfying $D((A + \lambda)^{-1}) = H$ for $|\arg\lambda| < \pi/2 + \omega$.

(A.6) $\|(A + \lambda)^{-1}x - (A + \lambda)^{-1}y\| \leqq (\sup\{\mathrm{Re}(e^{-i\theta}\lambda); |\theta| < \omega\})^{-1}\|x - y\|$ for $|\arg\lambda| \leqq \pi/2 + \omega$, $x, y \in H$.

(A.7) For any $x \in D(A)$ there exist a linear operator $\partial^2 A$ from H to $L(H; L(H; G))$ and a function $\varepsilon(\lambda) \geqq 0$ satisfying

$$\partial A(x + \lambda Ax + \varepsilon(\lambda))y = \partial A(x)y + \lambda \partial^2 A(x, Ax)y + \xi(\lambda)y, \tag{1.4}$$

$$\|\varepsilon(\lambda)\| \leqq E(\lambda) = o(|\lambda|), \qquad \|\xi(\lambda)\|_{L(H,G)} \leqq Z(\lambda) = o(|\lambda|^2).$$

Here $L(X; Y)$ is the space of linear operators from X to Y.

THEOREM 1. *For some complex sector $\Sigma_\theta = \{t \in \mathbf{C}; |\arg t| < \theta\}$, there exists a holomorphic function u which satisfies the following Cauchy problem*:

$$\text{(1.5)} \qquad \begin{cases} \dfrac{d}{dt}u(t) = Au(t), & t \in \Sigma_\theta, \\ u(0) = u_0 \in D(A). \end{cases}$$

2. Preliminaries

First, we cite a known theorem:

THEOREM A. *Let H be a Hilbert space and $-A$ be a maximal monotone operator from H to H. Then the Cauchy problem*

$$\text{(2.1)} \qquad \frac{d}{dt}u(t) = Au(t), \qquad 0 < t,$$

$$\text{(2.2)} \qquad u(0) = u_0 \in D(A),$$

has a unique solution $u(t)$.

REMARK. For properties of maximal monotone operators and the Cauchy problem (2.1) and (2.2), see Brezis [1] or Kōmura–Konishi [7].

LEMMA 1. *For any θ, $|\theta| < \omega$, $-e^{i\theta}A$ is a maximal monotone operator.*

PROOF. By virtue of (A.5) we have only to show the monotonicity of $-e^{i\theta}A$. By (A.6) we get

$$\|(\lambda + e^{i\theta}A)^{-1}x - (\lambda + e^{i\theta}A)^{-1}y\| = \|(e^{-i\theta}\lambda + A)^{-1}x - (e^{-i\theta}\lambda + A)^{-1}y\| \leqq (\mathrm{Re}(e^{-i\theta}\lambda))^{-1}\|x - y\|.$$

LEMMA 2. *Let $u(t)$ be a solution to (1.5). Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\text{(2.3)} \qquad u(s) = u_0 + sAu_0 + \varepsilon(s) \qquad \textit{for any } 0 < s < \delta$$

where $\|\varepsilon(s)\| \leqq C\varepsilon s$, C is a positive constant.

PROOF. By the relation

$$\frac{d^+}{dt}u(t)\big|_{u(t)=u_0} = Au_0,$$

we get (2.3).

LEMMA 3. *Let $u(t)$ be a solution to (1.5). Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$u(t) = u(s) + (t-s)Au(s) + \varepsilon(t-s) \qquad \text{for } 0 \leqq s < t, \quad |t-s| < \delta \tag{2.4}$$

where $\| \varepsilon(t-s) \| \leqq C\varepsilon |t-s|$.

PROOF. This follows from (A.4), (A.1) and

$$\left\| \frac{d^+}{dt} u(s) \right\| \leqq \left\| \frac{d^+}{dt} u(0) \right\| \qquad \text{for any } s > 0.$$

LEMMA 4. *For a solution $u(t)$ to (1.5), we have*

$$\begin{aligned} u(s) &= u_0 + \int_0^s Au(r)dr \\ &= u_0 + \int_0^s [Au_0 + r\partial A(u_0)Au_0 + \eta(r)]dr \qquad \text{for } s \leqq \alpha(u_0). \end{aligned} \tag{2.5}$$

PROOF. By Lemma 2 and (A.4) we have

$$Au(r) = A(u_0 + rAu_0 + \varepsilon(r)) = Au_0 + r\partial A(u_0)Au_0 + \eta(r).$$

Let $0 < \theta < \omega$ be fixed. By Lemma 1, there exists a unique solution $v^t(s)$ (resp. $\tilde{v}^t(s)$) to (2.6) and (2.7) (resp. (2.8) and (2.9)):

$$\frac{dv^t}{ds} \in e^{i\theta}Av^t(s), \qquad 0 < s \leqq T, \tag{2.6}$$

$$v^t(0) = u(t), \qquad 0 < t \leqq T \tag{2.7}$$

where $u(t)$ is a solution to (1.5);

$$\frac{d\tilde{v}^t}{ds} \in e^{-i\theta}A\tilde{v}^t(s), \qquad 0 < s \leqq T, \tag{2.8}$$

$$\tilde{v}^t(0) = u(t), \qquad 0 < t \leqq T. \tag{2.9}$$

Let

$$\begin{cases} u(0) = u_0, \\ u(t)\text{: solution to (1.5) for } 0 < t \leqq T, \\ u(z) = v^t(s), \quad \text{where } z = t + se^{i\theta}, \quad 0 < t, s \leqq T, \\ u(z) = \tilde{v}^t(s), \quad \text{where } z = t + se^{-i\theta}, \quad 0 < t, s \leqq T. \end{cases} \tag{2.10}$$

Let $\square_s = \square(t_0, s, \theta)$ be the parallelogram with vertices at t_0, $t_0 + s$, $t_0 + s + se^{i\theta}$ and $t_0 + se^{i\theta}$, and define

$$\int_{\square_s} f\,dz = \int_{t_0}^{t_0+s} f\,dz + \int_{t_0+s}^{t_0+s+se^{i\theta}} f\,dz + \int_{t_0+s+se^{i\theta}}^{t_0+se^{i\theta}} f\,dz + \int_{t_0+se^{i\theta}}^{t_0} f\,dz.$$

We shall show that $\int_{\square_s} u(z)dz = 0$.

Let $\Phi(t) = \int_0^t \eta(r)dr$ and $\tilde{\Phi}(t) = \int_0^t \Phi(r)dr$.

LEMMA 5. *There exist constants* $C_i > 0$ $(i = 1, \dots, 5)$ *such that*

$$\| \varepsilon(t) \| \leqq C_1 t, \quad \| \xi(t) \|_{L(H,G)} \leqq C_2 t^2, \quad \| \eta(t) \|_G \leqq C_3 t,$$
$$\| \Phi(t) \|_G \leqq C_4 t^2, \quad \| \tilde{\Phi}(t) \|_G \leqq C_5 t^3.$$

The proof is easy and omitted.

Let $u(t)$ be a solution to (2.1) and (2.2). Set

$$M_1 = \sup\{ \| u(t) \| ; 0 \leqq t \leqq T\},$$

$$M_2 = \sup\{ \| Au(t) \| ; 0 \leqq t \leqq T\},$$

$$M_3 = \sup\{ \| \partial A(u(t))Au(t) \|_G ; 0 \leqq t \leqq T\},$$

$$M_4 = \sup\{ \| (s-t)\partial^2 A(u(t), Au(t))Au(t) \|_{L(H,G)} ; 0 \leqq t < s \leqq T\}.$$

LEMMA 6. *Every constant* M_i $(i = 1, \dots, 4)$ *is finite.*

PROOF. Since $u(t)$ is a solution to (2.1) and (2.2), we have

$$\| Au(t) \| \leqq \| Au_0 \| \qquad \text{for any } 0 \leqq t.$$

This implies

$$M_2 = \| Au_0 \| \tag{2.11}$$

and

$$M_1 \leqq \| u_0 \| + T \| Au_0 \| . \tag{2.12}$$

By (A.4) we have $\| \partial A(u(t))Au(t) \|_G \leqq L \| Au(t) \|$. This implies

$$M_3 \leqq LM_2. \tag{2.13}$$

If $s > t$ and $s - t$ is sufficiently small, by (A.7) we get

$$\partial A(u(s))x = \partial A(u(t)) + (s-t)\partial^2 A(u(t), Au(t))x + \xi(s-t)x.$$

Then using (2.13) we get

$$\| (s-t)\partial^2 A(u(t), Au(t))x \|_G \leqq \| \partial A(u(s)) - \partial A(u(t) - \xi(s-t)) \|_{L(H,G)} \| x \|$$
$$\leqq (2LM_2 + C_2(s-t)^2) \| x \|$$
$$\leqq (2LM_2 + C_2 T^2) \| x \| .$$

Consequently we get

$$M_4 \leqq 2LM_2 + C_2 T^2. \tag{2.14}$$

3. Proof of Theorem 1

For simplicity we assume $t_0 = 0$. We abbreviate A_G to A and $\| \cdot \|_G$ to $\| \cdot \|$. Let

$$u(0) = u_0 \in D(A), \qquad u(s) = u(0) + \int_0^s Au(r)dr, \tag{3.1}$$

$$v(0) = u(t), \qquad v(s) = v(0) + \int_0^s e^{i\theta} Av(r)dr, \tag{3.2}$$

$$w(0) = u(0), \qquad w(s) = w(0) + \int_0^s e^{i\theta} Aw(r)dr, \tag{3.3}$$

$$P(0) = w(t), \qquad P(s) = P(0) + \int_0^s AP(r)dr, \tag{3.4}$$

and

$$\begin{gathered} \mathrm{I} = \int_0^t u(s)ds, \quad \mathrm{II} = \int_0^t v(s)ds, \quad \mathrm{III} = \int_0^s w(s)ds, \\ \mathrm{IV} = \int_0^t P(s)ds, \quad \mathrm{V} = \int_0^t [P(s) - v^s(t)]ds. \end{gathered} \tag{3.5}$$

LEMMA 7.

$$\begin{aligned} \mathrm{I} &= \int_0^t u(s)ds \\ &= tu_0 + \frac{t^2}{2} Au_0 + \frac{t^3}{6} \partial A(u_0) Au_0 + \tilde{\Phi}(t) \end{aligned} \tag{3.6}$$

where $\tilde{\Phi}(t) = \int_0^t \Phi(s)ds$ *and* $\Phi(s) = \int_0^s \eta(r)dr$.

PROOF. By (3.1) and Lemma 1 we get

$$
\begin{aligned}
u(s) &= u_0 + \int_0^s Au(r)dr \\
&= u_0 + \int_0^s [Au_0 + r\partial A(u_0)Au_0 + \eta(r)]dr \qquad (3.7)\\
&= u_0 + sAu_0 + \frac{s^2}{2}\partial A(u_0)Au_0 + \Phi(s).
\end{aligned}
$$

by integrating $u(s)$ from 0 to t we obtain (3.6).

LEMMA 8.

$$
\begin{aligned}
\mathrm{II} &= \int_0^t v(s)ds \\
&= tu_0 + t^2Au_0 + \frac{t^3}{2}\partial A(u_0)Au_0 + t\Phi(t) \\
&\quad + e^{i\theta}\frac{t^2}{2}(Au_0 + t\partial A(u_0)Au_0 + \eta(t)) \qquad (3.8)\\
&\quad + e^{2i\theta}\frac{t^3}{6}(\partial A(u_0) + t\partial^2 A(u_0, Au_0) + \xi(t)) \\
&\quad \times (Au_0 + t\partial A(u_0)Au_0 + \varepsilon(t)) + \tilde{\Phi}(t)e^{i\theta}.
\end{aligned}
$$

PROOF. By (3.1) and (3.2) we have

$$
\begin{aligned}
v(s) &= v(0) + \int_0^s e^{i\theta}Av(r)dr \qquad (3.9)\\
&= u_0 + \int_0^t Au(r)dr + \int_0^s e^{i\theta}Av(r)dr.
\end{aligned}
$$

By (A.4) and (A.7) we obtain

$$
\begin{aligned}
Av(r) &= A(v(0) + e^{i\theta}rAv(0) + \varepsilon(r)) \\
&= Av(0) + e^{i\theta}r\partial A(v(0))Av(0) + \eta(r) \\
&= Au_0 + t\partial A(u_0)Au_0 + \eta(t) \qquad (3.10)\\
&\quad + e^{i\theta}r(\partial A(u_0) + t\partial^2 A(u_0, Au_0) + \xi(t)) \\
&\quad \times (Au_0 + t\partial A(u_0)Au_0 + \varepsilon(t)) + \eta(r).
\end{aligned}
$$

Combining (3.10) and (3.9), (3.8) is easily obtained.

LEMMA 9.

$$\text{III} = \int_0^t w(s)ds \tag{3.11}$$
$$= tu_0 + e^{i\theta}\left(\frac{t^2}{2}Au_0 + e^{i\theta}\frac{t^3}{6}\partial A(u_0)Au_0 + \tilde{\Phi}(t)\right).$$

PROOF. From (3.1) and (3.3) it follows that

$$w(s) = w(0) + \int_0^s e^{i\theta}Aw(r)dr = u_0 + \int_0^s e^{i\theta}Aw(r)dr. \tag{3.12}$$

From (A.4), (3.3) and Lemma 2 it follows that

$$Aw(r) = A(w(0) + e^{i\theta}rAu_0 + \varepsilon(r)) \tag{3.13}$$
$$= Au_0 + e^{i\theta}r\partial A(u_0)Au_0 + \eta(r).$$

(3.12) and (3.13) imply

$$w(s) = u_0 + e^{i\theta}\left(sAu_0 + e^{i\theta}\frac{s^2}{2}\partial A(u_0)Au_0 + \Phi(s)\right). \tag{3.14}$$

By integrating $w(s)$ from 0 to t we obtain (3.11).

LEMMA 10.

$$\text{IV} = \int_0^t P(s)ds$$
$$= tu_0 + \frac{t^2}{2}((Au_0 + \eta(t)) + \xi(t))$$
$$+ \frac{t^3}{6}(\partial A(u_0)(Au_0 + \eta(t)) + \xi(t)(Au_0 + \eta(t)))$$
$$+ e^{i\theta}\left(t^2Au_0 + t\Phi(t) + \frac{t^3}{2}\partial A(u_0)Au_0 + \frac{t^4}{6}\partial^2 A(u_0, Au_0)(Au_0 + \eta(t))\right. \tag{3.15}$$
$$\left. + \frac{t^4}{6}\partial A(u_0)\partial A(u_0)Au_0 + \xi(t)\frac{t^4}{6}\partial A(u_0)Au_0\right)$$
$$+ e^{2i\theta}\left(\frac{t^3}{2}\partial A(u_0)Au_0 + \frac{t^5}{6}\partial^2 A(u_0, Au_0)\partial A(u_0)Au_0\right).$$

PROOF. By (3.4) we get

$$P(s) = P(0) + \int_0^s AP(r)dr = w(t) + \int_0^s AP(r)dr. \tag{3.16}$$

From (3.12) it follows that

$$P(0) = u_0 + e^{i\theta}\left(tAu_0 + e^{i\theta}\frac{t^2}{2}\,\partial A(u_0)Au_0 + \Phi(t)\right). \tag{3.17}$$

By Lemma 2 and (A.4) we get

$$\begin{aligned} AP(r) &= A(P(0) + r\partial AP(0) + \varepsilon(r)) \\ &= Aw(t) + r\partial A(w(t))Aw(t) + \eta(r). \end{aligned} \tag{3.18}$$

By (3.13) we have

$$Aw(r) = Au_0 + e^{i\theta}t\partial A(u_0)Au_0 + \eta(t). \tag{3.19}$$

By (A.7) we obtain

$$\begin{aligned} Aw(t) &= \partial A(u_0 + e^{i\theta}tAu_0 + \varepsilon(t)) \\ &= \partial A(u_0) + e^{i\theta}t\partial^2 A(u_0, Au_0) + \xi(t). \end{aligned} \tag{3.20}$$

Then from (3.16), (3.17), (3.18), (3.19) and (3.20) we get

$$\begin{aligned} P(s) = u_0 &+ e^{i\theta}\left(tAu_0 + e^{i\theta}\frac{t^2}{2}\,\partial A(u_0)Au_0 + \Phi(t)\right) \\ &+ s(Au_0 + e^{i\theta}t\partial A(u_0)Au_0 + \eta(t)) \\ &+ \frac{s^2}{2}((\partial A(u_0) + e^{i\theta}t\partial^2 A(u_0, Au_0) + \xi(t)) \\ &\qquad \times (Au_0 + e^{i\theta}t\partial A(u_0)Au_0 + \eta(t)) + s\xi(t)). \end{aligned} \tag{3.21}$$

By integrating $P(s)$ from 0 to t we have (3.15).

LEMMA 11.

$$V = \int_0^t [P(s) - v^s(t)]ds$$

$$= \frac{t^4}{12} e^{i\theta}(\partial A(u_0)\partial A(u_0)Au_0 + \partial^2 A(u_0, Au_0)Au_0)$$

$$(3.22) \qquad - \tilde{\Phi}(t) + \frac{t^2}{2}\eta(t) + \frac{t^2}{2}\xi(t) + \frac{t^3}{6}\xi(t)(Au_0 + e^{i\theta}t\partial A(u_0)Au_0 + \eta(t))$$

$$- \frac{t^2}{2} e^{i\theta} \int_0^t [(\partial A(u_0) + s\partial^2 A(u_0, Au_0))\varepsilon(s)$$

$$+ \xi(s)(Au_0 + s\partial A(u_0)Au_0) + \varepsilon(s)\xi(s)]ds.$$

PROOF. By (2.6), (2.7), (3.1), (A.4) and Lemma 2 we have

$$v^s(t) = u(s) + \int_0^t e^{i\theta} A v^s(r)dr$$

$$= u_0 + sAu_0 + \frac{s^2}{2}\partial A(u_0)Au_0 + \Phi(s)$$

$$+ e^{i\theta} \int_0^t [Au_0 + s\partial A(u_0)Au_0 + \eta(s)$$

$$+ e^{i\theta} r(\partial A(u_0) + s\partial^2 A(u_0, Au_0) + \xi(s))$$

$$\times (Au_0 + s\partial A(u_0)Au_0 + \varepsilon(s)) + \eta(r)]dr$$

$$(3.23) \qquad = u_0 + sAu_0 + \frac{s^2}{2}\partial A(u_0)Au_0 + \Phi(s)$$

$$+ e^{i\theta}(tAu_0 + st\partial A(u_0)Au_0 + t\eta(s))$$

$$+ \frac{t^2}{2} e^{i\theta}(\partial A(u_0)Au_0 + s\partial A(u_0)(\partial A(u_0)A(u_0))$$

$$+ s\partial^2 A(u_0, Au_0)Au_0 + s^2\partial^2 A(u_0, Au_0)(\partial A(u_0)Au_0)$$

$$+ (\partial A(u_0) + s\partial^2 A(u_0, Au_0) + \xi(s))\varepsilon(s)$$

$$+ \xi(s)(Au_0 + s\partial A(u_0)Au_0 + \varepsilon(s))) + \Phi(t)\varepsilon^{2i\theta}.$$

Then by (3.21) and (3.23) we obtain

$$\int_0^t [P(s) - v^s(t)]ds$$

$$= \int_0^t \Big[e^{i\theta}\Phi(t) + s\eta(t) + \frac{s^2}{2}\xi(t)(Au_0 + e^{i\theta}t\partial A(u_0)Au_0 + \eta(t))$$

$$+ s\xi(t) - \Phi(s) - e^{i\theta}t\eta(s)$$

$$- e^{i\theta}\frac{t^2}{2}((\partial A(u_0) + s\partial^2 A(u_0, \partial A(u_0)))\varepsilon(s)$$

$$+ \xi(s)(Au_0 + s\partial A(u_0)Au_0) + \varepsilon(s)\xi(s)$$

$$+ (s-t)\frac{st}{2}e^{i\theta}(\partial A(u_0)\partial A(u_0)Au_0 + \partial^2 A(u_0, Au_0)Au_0))\Big] ds \tag{3.24}$$

$$= -\tilde{\Phi}(t)\frac{t^2}{2}\eta(t) + \frac{t^3}{6}\xi(t)(Au_0 + e^{i\theta}t\partial A(u_0)Au_0 + \eta(t))$$

$$+ \frac{t^2}{2}\xi(t) - e^{i\theta}\frac{t^2}{2}\int_0^t [(\partial A(u_0) + s\partial A^2(u_0, Au_0))\varepsilon(s)$$

$$+ \xi(s)(Au_0 + s\partial A(u_0)Au_0) + \varepsilon(s)\xi(s)]ds$$

$$+ \frac{t^4}{12}e^{i\theta}(\partial A(u_0)\partial A(u_0)Au_0 + \partial^2 A(u_0, Au_0)Au_0).$$

Thus (3.22) is obtained.

LEMMA 12. *For sufficiently small $t > 0$ we have*

$$\left\| \int_{\square_t} u(z)dz \right\| \leqq Ct^3.$$

PROOF. From (2.6), ..., (2.10) and (3.1), ..., (3.5) it follows that

$$\int_{\square_t} u(z)dz = \int_0^t u(s)ds + \int_0^t v(s)e^{i\theta}ds - \int_0^t v^s(t)ds - \int_0^t w(s)e^{i\theta}ds$$

$$= \int_0^t u(s)ds + \int_0^t v(s)e^{i\theta}ds - \int_0^t P(s)ds$$

$$- \int_0^t w(s)e^{i\theta}ds + \int_0^t [P(s) - v^s(t)]ds$$

$$= \mathrm{I} + e^{i\theta}\mathrm{II} - e^{i\theta}\mathrm{III} - \mathrm{IV} + \mathrm{V}.$$

Then by Lemma 7, ..., Lemma 11 we get

$$\int_{\square_t} u(z)dz = \frac{1}{6}t^4(-e^{i\theta}(\partial A(u_0)\partial A(u_0)Au_0 - \partial^2 A(u_0, Au_0)Au_0)$$

$$- e^{2i\theta}t\partial^2 A(u_0, Au_0)\partial A(u_0)Au_0 + te^{3i\theta}\partial^2 A(u_0, Au_0)\partial A(u_0)Au_0)$$

$$-\frac{1}{6}t^3\partial A(u_0)\eta(t)$$

$$+ e^{i\theta}\Bigg(t\Phi(t) - \frac{t^4}{6}\partial^2 A(u_0, Au_0)\eta(t)$$

$$-\frac{t^2}{2}\int_0^t [(\partial A(u_0) + s\partial^2 A(u_0, Au_0))\varepsilon(s)$$

$$+ \xi(s)(Au_0 + s\partial A(u_0)Au_0) + \varepsilon(s)\xi(s)]ds\Bigg)$$

$$+ e^{2i\theta}\frac{t^2}{2}\eta(t)$$

$$+ e^{3i\theta}\frac{t^3}{6}((\partial A(u_0) + t\partial^2 A(u_0, Au_0))\varepsilon(t)$$

$$+ \xi(t)(Au_0 + t\partial A(u_0)Au_0) + \xi(t)\varepsilon(t)).$$

Then from Lemma 5 there exists a positive constant C such that

$$\left\| \int_{\square_t} u(z)dz \right\| \leqq Ct^3.$$

There exists $n \in \mathbf{N}_+$ such that

$$2^{-n}T < \inf\{\varepsilon(u(t)); 0 \leqq |t| < T\}.$$

For $t = 2^{-n}s$, we denote $\square(jt + kte^{i\theta}, t, \theta)$ by $\square_{j,k,t}$. By Lemma 12 we obtain

$$\left\| \int_{\square_s} u(z)dz \right\|_G \leqq \sum_{j,k=1}^{2^n} \left\| \int_{\square_{j,k,t}} u(z)dz \right\|_G \leqq C4^n t^3 = C4^n\left(\frac{s}{2^n}\right)^3 \leqq C\frac{T^3}{2^n}.$$

Therefore for any $\varepsilon > 0$ there exists large n such that

$$\left\| \int_{\square_s} u(z)dz \right\|_G \leqq C\frac{T^3}{2^n} < \varepsilon.$$

It means that

$$\int_{\square_s} u(z)dz = 0.$$

In the case that $-\omega < \theta < 0$, the proof is analogous to that of $0 < \theta < \omega$. Hence $u(z)$ is analytic.

To show that u is unique, it suffices to show it for real t since u is analytic. However, for real t, uniqueness of u is included in Theorem A.

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